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General Three-Dimensional Wave Propagation in Nonequilibrium Magnetogasdynamics

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1. INTRODUCTION

With the advancement of the science of high speed flights at very high altitudes it has become necessary to include into the conventional gasdynamical approach the effects due to varying the chemical composition (or the vibrational relaxation) of the gas.

The analytical account of these new effects necessitates, in addition to the two usual thermodynamical variables, the introduction of another thermophysical variable q (which may be termed as the relaxation variable or internal vibrational temperature) in the state of the fluid. The addition of this new independent thermodynamical parameter q into the fluid system leads to a problem that is quite different from the usual one. In fact a new equation termed as the 'rate equation' is added to the system of relations for the conventional gasdynamics. But this new system of equations turns out to be highly nonlinear (that is, not quasilinear). Thus, the usual analytical approaches to the flow problems do not make themselves amenable to mathematical calculations. Accordingly, the approach of small perturbation theory was utilized by Broer [1] and Vincenti [2]. Confining his interest to an inviscid compressible nonheat conducting gas, Broer has shown that there exist, corresponding to high and low frequencies, two limiting speeds in chemically reacting fluids. Vincenti (using Broer's rate equation) has obtained a small-disturbance solution for the steady two-dimensional flow over a sinusoidal wall of an inviscid gas in vibrational or chemical nonequilibrium. His results illustrate in a simple fashion some of the properties of nonequilibrium flow such as the occurence of pressure drag at subsonic speeds and the absence of the discontinuous phenomena that characterize the Prandtl-Glauert theory when the flow changes from subsonic to supersonic. Later on, Stupochenko and Stakhanov [3], using a different rate equation, arrived at a result similar to that obtained by Broer for the problem of wave propagation. These studies became responsible for further development [4, 5] and its generalization to relaxation magnetohydrodynamics (cf. Coburn [6] and Bhutani [7, 8]). Applying discontinuity theory (or the corresponding theory of the Cauchy initial value problem [9]), Coburn, using the rate equation of [3], has obtained the limiting speeds in the nonlinear theory for characteristic waves in charged compressible relaxation hydrodynamics. In [7], after deriving under the linearized approximation (using Broer's rate equation), a single equation governing the plane steady flows that depart from equilibrium and are subjected to an aligned magnetic field, we have obtained a number of interesting results concerning the stability of such a motion, the wave angle of the disturbance, and the diffusion coefficient. In particular, we have shown that the region of influence is increased with an increase in the value of the magnetic parameter m; the nonequilibrium state of the gas gives rise to diffusion effect, and the diffusion coefficient increases with m. In [8], we have extended (using Broer's rate equation) the work of Vincenti and Whitham [10] to nonequilibrium magnotogasdynamics. The last mentioned study has clarified the role of the nonequilibrium parameter.

Recently Yuan [11],[1] using a very general rate equation, has extended the work of Broer, Stupochenko, and Stakhanov to the case of heat conducting fluids. More specifically he has shown that the limiting speed of the low frequency wave remains the same as in the nonheat conduction case, but due to heat conduction the speed of the high frequency wave is changed.

We have in this paper (using Broer's rate equation) taken up the problem of propagation of small disturbances in general three-dimensional unsteady nonequilibrium magnetogasdynamics. Assuming an exponential solution of the type $\exp [2\pi/\lambda(\mathbf{n} \cdot \mathbf{r} - v_0 t)]$, where λ, $\mathbf{n}$, v_0, etc. are defined in Section 3, we have derived a fifth degree algebraic equation for v_0 and obtained its explicit solution for the particular cases of (i) no magnetic field and; (ii) a transverse magnetic field.

For these particular cases, as well as the general case, it turns out that one root corresponds to pure decay. The remaining four roots in the general case give the damping characteristic of slow and fast waves. For showing the influence of the various parameters on the propagation properties in the general case, we have drawn the velocity phase diagram for different values of the parameter $\delta = k^2 c_A{}^2$, where k is the relaxation parameter and c_A is Alfvén's speed. The interesting conclusions that result from this diagram are summed up in Section 5.

It may be stressed here that this diagram, when used in full appreciation of the limitations involved, becomes a very good aid for visualizing and understanding the basic phenomena underlying wave propagation through

[1] We are grateful to Prof. Coburn for drawing our attention and making available to us the work of C. Yuan [11].

relaxing media. Unexpectedly, this diagram comes out to be almost the same as in the case of the Hall effect on wave propagation [12].

Further, for showing the influence of one set of waves (obtained when there is relaxation) on the other (corresponding to the equilibrium case) we have, using Whitham's technique [10], obtained results concerning the stability of such a motion, the decaying factor, and the diffusion coefficient of the waves. In particular, we have shown that chemical reactions give rise to diffusion effects, and the diffusion coefficient increases with the increase in the strength of the magnetic parameter.

Finally, in Appendix A of this paper we have shown how our scheme of working with enthalpy instead of entropy (which is usually used by engineers and also used by Coburn and Yuan) is related to Yuan's scheme. Further in Appendix B, we have discussed the position of the rate equation used by Broer (also used in the present paper) to that of Stupochenko and Stakhanov.

2. Derivation of the Fundamental Equations

As pointed out by Coburn [6] and also justified in [7], in the study of the motion of infinitely electrically conducting gases that depart from equilibrium, the basic equations of the system consist of the usual equations of continuity, motion, and the Maxwell equations. But the usual energy equation is replaced by two equations. One of these equations is the usual energy equation with or without an additional term, depending on whether the equation is taken in terms of the entropy s or the enthalpy h. In case the equation taken involves entropy then the additional term involves a new dependent variable q, called the relaxation variable. If the energy equation is taken in terms of enthalpy h, then there is no additional term involved but q is involved through the equation of state

$$h = h(p, \rho, q). \tag{2.1}$$

Equation (2.1) states that unlike the gasdynamical case, enthalpy h is also a function of q, the relaxation variable. It should be noted that the above remarks are valid when the fluid is *nonheat conducting*. For the case of a heat conducting compressible magnetic (infinitely conductive fluid) see Eq. A.4 of Appendix A.

The second equation, which is a supplement to the energy equation, gives the rate of change of q along the stream lines and is termed as the rate equation. We shall use the rate equation (see (2.6)) of Broer [1]. For a more general rate equation, see the work of C. Yuan [11] and Appendix B of this paper.

Keeping the above remarks in view, we give below the fundamental equations (in linearized form) governing the general three-dimensional wave propagation in nonequilibrium magnetogasdynamics of perfect electrical conductors (for Eqs. (2.2)-(2.5) see Banös [13]).

$$\frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot \mathbf{V} = 0, \tag{2.2}$$

$$\rho_0 \frac{\partial \mathbf{V}}{\partial t} = - \nabla p + \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B}_0 , \tag{2.3}$$

$$\frac{\partial \mathbf{B}}{\partial t} = - \nabla \times (\mathbf{V} \times \mathbf{B}_0), \tag{2.4}$$

$$\frac{\partial h}{\partial t} = \frac{1}{\rho_0} \frac{\partial p}{\partial t}, \tag{2.5}$$

where ρ, $\mathbf{V}, p$, $\mathbf{B}$, h are, respectively, the density, velocity vector of the gas, pressure, magnetic induction, and enthalpy of the gas in the perturbed state and the subscript zero refers to the unperturbed state. Broer's equation becomes

$$\tau_0 \frac{\partial q}{\partial t} = \bar{q} - q. \tag{2.6}$$

In Eq. (2.6), τ_0 is the relaxation time of the nonequilibrium process evaluated at the initial equilibrium state and $\bar{q}(p, \rho) = q_0$ is the initial equilibrium value of q. It may be remarked here that Eq. (2.6) results from a power series expansion of the chemical affinity and the neglect of all terms of higher than the first order (cf. C. Yuan [11] or the Appendix B).

Equation (2.1) on linearization gives

$$dh = h_{p_0}\, dp + h_{\rho_0}\, d\rho + h_{q_0}\, dq, \tag{2.7}$$

where h_{p_0} denotes $[(\partial h/\partial p)_{\rho,q}]_0$, the initial equilibrium state, and similarly for h_{ρ_0} and h_{q_0}.

Since q_0 denotes the value of q at the equilibrium state or $q_0 = \bar{q}(p, \rho)$, we get on differentiation and linearization

$$d\bar{q} = \bar{q}_{p_0}\, dp + \bar{q}_{\rho_0}\, d\rho. \tag{2.8}$$

Equations (2.2)-(2.8) are eleven equations for the eleven variables ρ, $\mathbf{V}$, $\mathbf{B}$, p, h, q, $\bar{q}$.

Differentiating (2.6) with respect to time and eliminating $\partial\bar{q}/\partial t$ by use of (2.8), we find

$$\left(1+\tau_0\frac{\partial}{\partial t}\right)\frac{\partial q}{\partial t}=\bar{q}_{p_0}\frac{\partial p}{\partial t}+\bar{q}_{\rho_0}\frac{\partial \rho}{\partial t}. \tag{2.9}$$

Further, Eqs. (2.5) and (2.7) yield

$$\left(\frac{1}{\rho_0}-h_{p_0}\right)\frac{\partial p}{\partial t}=h_{\rho_0}\frac{\partial \rho}{\partial t}+h_{q_0}\frac{\partial q}{\partial t}. \tag{2.10}$$

Thus the system of equations that we have to consider reduces to Eqs. (2.2)-(2.4), (2.9), and (2.10) for the nine unknowns ρ, q, p, $\mathbf{V}$, $\mathbf{B}$.

3. Derivation of the Dispersion Equation

If we assume that the displacements $\xi(r, t)$, given by $\partial\xi/\partial t = \mathbf{V}$, and ρ, q, p, $\mathbf{B}$, all vanish at some time $t = 0$, then the equations (2.2), (2.4), (2.9), and (2.10) can be integrated to give, respectively, the following equations

$$\rho+(\rho_0\nabla\cdot\xi)=0, \tag{3.1}$$

$$\mathbf{B}=-\nabla\times(\xi\times\mathbf{B}_0), \tag{3.2}$$

$$\left(1+\tau_0\frac{\partial}{\partial t}\right)q=\bar{q}_{p_0}p+\bar{q}_{\rho_0}\rho, \tag{3.3}$$

$$\left(\frac{1}{\rho_0}-h_{p_0}\right)p=h_{\rho_0}\rho+h_{q_0}q. \tag{3.4}$$

To determine the differential equation for ξ, we need the value of $(\nabla\times\mathbf{B})\times\mathbf{B}_0$. Using Eq. (3.2) we can write

$$(\nabla\times\mathbf{B})\times\mathbf{B}_0=\{\nabla[B_0^2\nabla\cdot\xi-(\mathbf{B}\cdot\nabla)(\mathbf{B}_0\cdot\xi)]+(\mathbf{B}_0\cdot\nabla)^2\xi-\mathbf{B}_0[\mathbf{B}_0\cdot\nabla(\nabla\cdot\xi)]\}. \tag{3.5}$$

Eliminating q and ρ in (3.3) by use of (3.1), (3.4), we obtain

$$\left[\tau_0\left(\frac{1}{\rho_0}-h_{p_0}\right)\frac{\partial}{\partial t}+\left(\frac{1}{\rho_0}-h_{p_0}-h_{q_0}\bar{q}_{p_0}\right)\right]p$$
$$=-\rho_0\left[h_{\rho_0}\tau_0\frac{\partial}{\partial t}+(h_{\rho_0}+h_{q_0}\bar{q}_{\rho_0})\right]\nabla\cdot\xi. \tag{3.6}$$

Introducing the limiting speeds for high and low frequency (see Broer [1] and Vincenti [2]), that is, defining by

$$c_f^2 = \frac{h_{\rho_0}}{\dfrac{1}{\rho_0} - h_{p_0}}, \tag{3.7}$$

$$c_e^2 = \frac{h_{\rho_0} + h_{q_0}\bar{q}_{\rho_0}}{\dfrac{1}{\rho_0} - h_{p_0} - h_{q_0}\bar{q}_{p_0}}, \tag{3.8}$$

where c_f and c_e denote the frozen sound speed and the equilibrium speed, respectively, the Eq. (3.6) reduces to

$$\left[c_f^2 + c_e^2 k_0 \frac{\partial}{\partial t}\right] p = -\rho_0 c_e^2 c_f^2 \left[1 + k_0 \frac{\partial}{\partial t}\right] \nabla \cdot \boldsymbol{\xi}, \tag{3.9}$$

where k_0 is the relaxation parameter defined by

$$k_0 = \frac{h_{\rho_0}\tau_0}{(h_{\rho_0} + h_{q_0}\bar{q}_{\rho_0})}. \tag{3.10}$$

Equations (3.5) and (3.9), on substitution into (2.3), give

$$\begin{aligned}\left[c_f^2 + k_0 c_e^2 \frac{\partial}{\partial t}\right] \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} - c_e^2 c_f^2 \left[1 + k_0 \frac{\partial}{\partial t}\right] \nabla(\nabla \cdot \boldsymbol{\xi}) \\ - \left[k_0 c_e^2 c_A^2 \frac{\partial}{\partial t} + c_A^2 c_f^2\right] \\ \times [\nabla\{\nabla \cdot \boldsymbol{\xi} - (\mathbf{b} \cdot \nabla)(\mathbf{b} \cdot \boldsymbol{\xi})\} \\ + (\mathbf{b} \cdot \nabla)^2 \boldsymbol{\xi} - \mathbf{b}\{\mathbf{b} \cdot \nabla(\nabla \cdot \boldsymbol{\xi})\}] = 0,\end{aligned} \tag{3.11}$$

where $\mathbf{b}$ is the unit vector along $\mathbf{B}_0$ and c_A is the Alfvén speed given by $c_A^2 = B_0^2/4\pi\rho_0$.

To obtain the dispersion relation, we assume that equation (3.11) has a plane wave solution of the type

$$\boldsymbol{\xi}(r, t) = \boldsymbol{\xi}_0 \exp\left[\frac{2\pi i}{\lambda}(\mathbf{n} \cdot \mathbf{r} - v_0 t)\right], \tag{3.12}$$

where λ is the wave length, $\mathbf{n}$ is a unit vector along the direction of propagation and $\boldsymbol{\xi}_0$ is the wave amplitude.

In Eq. (3.12), the real part of v_0 gives the phase velocity and the imaginary part shows the growth rate of the amplitude.

Using Eq. (3.12) in (3.11), we get

$$[c_f^2 - k'v_0c_e^2]\, v_0^2\xi - [(\omega_0^2c_f^2 - k'v_0c_e^2\omega^2)\,\mathbf{n}\cdot\xi - (c_f^2 - c_e^2k'v_0)\, c_A^2\cos^2\theta(\mathbf{b}\cdot\xi)]\,\mathbf{n} - c_A^2(c_f^2 - c_e^2k'v_0)\,[\cos^2\theta\xi - \cos\theta(\mathbf{n}\cdot\xi)\,\mathbf{b}] = 0, \quad (3.13)$$

where

$$k' = \frac{2\pi i}{\lambda} k_0, \qquad \omega^2 = c_A^2 + c_f^2, \qquad \omega_0^2 = c_e^2 + c_A^2, \quad (3.14)$$

and θ is the angle between the direction of magnetic field and direction of wave propagation. Further, we may write

$$|\,\xi\,|\cos\theta = \xi\cdot\mathbf{b}. \quad (3.14a)$$

To obtain the relations between v_0, k, c_e, c_f, which are a consequence of Eqs. (3.13), we form the scalar products of (3.13) with $\mathbf{n}$, $\mathbf{b}$, $\mathbf{n}\times\mathbf{b}$ and obtain

$$[(c_f^2 - k'v_0c_e^2)\, v_0^2 + k'v_0c_e^2\omega^2 - c_f^2\omega_0^2]\,\mathbf{n}\cdot\xi + c_A^2\cos\theta[c_f^2 - k'v_0c_e^2]\,\mathbf{b}\cdot\xi = 0, \quad (3.15)$$

$$v_0^2[c_f^2 - k'v_0c_e^2]\,\mathbf{b}\cdot\xi - \cos\theta[c_f^2\omega_0^2 - k'v_0c_e^2\omega^2 - c_A^2(c_f^2 - k'v_0c_e^2)]\,\mathbf{n}\cdot\xi = 0, \quad (3.16)$$

$$[c_f^2 - c_e^2k'v_0]\,[v_0^2 - c_A^2\cos^2\theta]\,\mathbf{b}\cdot(\mathbf{n}\times\xi) = 0. \quad (3.17)$$

From Eq. (3.17), we get the usual Alfvén mode $v_0^2 = c_A^2\cos^2\theta$, and due to nonequilibrium effect, the purely attenuated mode, i.e., $v_0 = c_f^2/c_e^2k'$.

By eliminating $\mathbf{n}\cdot\xi$ in (3.15), (3.16), and using (3.14a), we obtain the dispersion relation

$$k'c_e^2v_0^5 - c_f^2v_0^4 - k'c_e^2\omega^2v_0^3 + \omega_0^2c_f^2v_0^2 + c_A^2\cos^2\theta k'c_e^2c_f^2v_0 - c_e^2c_f^2c_A^2\cos^2\theta = 0. \quad (3.18)$$

To obtain a necessary check on our calculations we consider (3.18) in the equilibrium case by putting $k'\to 0$ and that gives

$$v_0^4 - \omega_0^2v_0^2 + c_A^2c_e^2\cos^2\theta = 0. \quad (3.19)$$

Corresponding to positive and negative signs of roots of v_0^2, Eq. (3.19) gives rise to fast and slow waves [14].

As it is not possible to solve Eq. (3.18) for all k', we shall discuss some special cases in Section 4.

4. Particular Cases of (3.18)

Case I. When $c_A = 0$. For the nonmagnetic and nonequilibrium case, Eq. (3.18) reduces to (as $\omega^2 = c_f^2$, $\omega_0^2 = c_e^2$ by (3.14)).

$$v_0^3 + \frac{i\beta}{k\alpha} v_0^2 - \beta v_0 - \frac{i\beta}{k} = 0, \tag{4.1}$$

where

$$k' = ik, \qquad c_e^2 = \alpha, \qquad c_f^2 = \beta. \tag{4.1a}$$

Under the condition $k^2 > (\beta/3\alpha^2)$, the roots of Eq. (4.1) can be put in the form (where δ_1 is defined by (4.2))

$$v_1 = -\frac{i\left[2\sqrt{\beta(3\alpha^2k^2 - \beta)}\sinh\frac{\phi}{3} + \beta\right]}{3\alpha k} = -i\delta_1, \tag{4.2}$$

$$v_2 = \left[\frac{\beta}{k\delta_1} - \frac{\left\{\sqrt{\beta(3\alpha^2k^2 - \beta)}\sinh\frac{\phi}{3} - \beta\right\}^2}{9\alpha^2k^2}\right]^{1/2} + i\,\frac{\sqrt{\beta(3\alpha^2k^2 - \beta)}\sinh\frac{\phi}{3} - \beta}{3\alpha k}, \tag{4.3}$$

$$v_3 = -\left[\frac{\beta}{k\delta_1} - \frac{\left\{\sqrt{\beta(3\alpha^2k^2 - \beta)}\sinh\frac{\phi}{3} - \beta\right\}^2}{9\alpha^2k^2}\right]^{1/2} + i\,\frac{\sqrt{\beta(3\alpha^2k^2 - \beta)}\sinh\frac{\phi}{3} - \beta}{3\alpha k}, \tag{4.4}$$

where

$$\sinh\phi = \frac{2\beta^2 - 9\beta\alpha^2k^2 + 27k^2\alpha^3}{2(3\alpha^2k^2 - \beta)\sqrt{\beta(3\alpha^2k^2 - \beta)}}. \tag{4.5}$$

Similarly for the case $k^2 < \beta/3\alpha^2$, the roots of (4.1) are

$$v_1 = -i\,\frac{\left[2\sqrt{\beta(\beta - 3\alpha^2k^2)}\cosh\frac{\phi}{3} + \beta\right]}{3\alpha k} = -i\delta_2, \tag{4.6}$$

$$v_2 = \left[\frac{\beta}{k\delta_2} - \frac{\left\{\sqrt{\beta(\beta - 3\alpha^2k^2)}\cosh\frac{\phi}{3} - \beta\right\}^2}{9\alpha^2k^2}\right]^{1/2} + i\,\frac{\sqrt{\beta(\beta - 3\alpha^2k^2)}\cosh\frac{\phi}{3} - \beta}{3\alpha k}, \tag{4.7}$$

$$v_3 = -\left[\frac{\beta}{k\delta_2} - \frac{\left\{\sqrt{\beta(\beta - 3\alpha^2k^2)}\cosh\frac{\phi}{3} - \beta\right\}^2}{9\alpha^2k^2}\right]^{1/2} + i\,\frac{\sqrt{\beta(\beta - 3\alpha^2k^2)}\cosh\frac{\phi}{3} - \beta}{3\alpha k}, \tag{4.8}$$

where

$$\cosh\phi = \frac{2\beta^2 - 9\beta\alpha^2k^2 + 27\alpha^3k^2}{2(\beta - 3\alpha^2k^2)\sqrt{\beta(\beta - 3\alpha^2k^2)}}. \tag{4.9}$$

From Eqs. (4.2)-(4.4) and (4.6)-(4.8), it is clear that out of the three compression modes given either under the condition $k^2 > \beta/3\alpha^2$ or $k^2 < \beta/3\alpha^2$ one of them i.e., v_1 is highly attenuated and the remaining two modes are propagated in opposite directions. The attenuation of the modes depend on the parameter k.

CASE II. Transverse Magnetic Field, $\theta = \pi/2$.

For the case under consideration, the Eq. (3.18) reduces to

$$v_0^3 + \frac{i\beta}{\alpha k}v_0^2 - \omega^2 v_0 - \frac{i\beta\omega_0^2}{\alpha k} = 0. \tag{4.10}$$

Under the condition $k^2 < \beta^2/3\alpha^2\omega^2$ the roots of (4.10) are given by (where δ_3 is defined by (4.11))

$$v_1 = -i\,\frac{\left[2\sqrt{(\beta^2 - 3\alpha^2k^2\omega^2)}\cosh\frac{\phi}{3} + \beta\right]}{3\alpha k} = -i\delta_3, \tag{4.11}$$

$$v_2 = \left[\frac{\beta\omega_0^2}{\alpha k\delta_3} - \frac{\left\{\sqrt{(\beta^2 - 3\alpha^2k^2\omega^2)}\cosh\frac{\phi}{3} - \beta\right\}^2}{9\alpha^2k^2}\right]^{1/2} + i\,\frac{\sqrt{(\beta^2 - 3\alpha^2k^2\omega^2)}\cosh\frac{\phi}{3} - \beta}{3\alpha k}, \tag{4.12}$$

$$v_3 = -\left[\frac{\beta\omega_0^2}{\alpha k\delta_3} - \frac{\left\{\sqrt{(\beta^2 - 3\alpha^2k^2\omega^2)}\cosh\frac{\phi}{3} - \beta\right\}^2}{9\alpha^2k^2}\right]^{1/2}$$

$$+ i\,\frac{\sqrt{(\beta^2 - 3\alpha^2k^2\omega^2)}\cosh\frac{\phi}{3} - \beta}{3\alpha k}. \tag{4.13}$$

Similar results apply for the case $k^2 > \beta^2/3\alpha^2\omega^2$.

The nature of the roots of (4.10) is quite obvious from Eqs. (4.11)-(4.13). The electromagnetic effects appear through the parameter ω_0 and ω and relaxation effects through k (see (4.1a)).

CASE III. When $c_f^2/c_e^2 = 1 + \epsilon$.

To obtain the roots of Eq. (3.18) for all values of the parameter k and the angle θ, we use the approximation $c_f^2/c_e^2 = 1 + \epsilon$, where ϵ is such that its second and higher power can be neglected. Accordingly, we get

$$\begin{aligned}&(v_0k' - 1)(v_0^4 - \omega_0^2v_0^2 + c_A^2c_e^2\cos^2\theta)\\ &= \epsilon(v_0^2 + c_A^2\cos^2\theta)\,c_e^2k'v_0 + \epsilon(v_0^4 - \omega_0^2v_0^2 + c_e^2c_A^2\cos^2\theta).\end{aligned} \tag{4.14}$$

It can be seen that out of five roots of the above equation one corresponds to pure decay and the other four give the damping characteristics of slow and fast waves.

Since we are mainly interested in the modification of the magneto-sonic modes due to the relaxation effects, we write (4.14) in the following form:

$$\begin{aligned}&(v_0^4 - \omega_0^2v_0^2 + c_A^2c_e^2\cos^2\theta)\\ &\approx -\frac{\epsilon(iv_0k + 1)}{(1 + k^2v_0^2)}\left[(v_0^2 + c_A^2\cos^2\theta)\,c_e^2ikv_0 + v_0^4 - \omega_0^2v_0^2 + c_A^2c_e^2\cos^2\theta\right],\end{aligned} \tag{4.15}$$

where $ik = k'$.

Using Eq. (3.19) in the form

$$v_0^2 = \frac{\omega_0^2 \pm \sqrt{(\omega_0^4 - 4c_e^2c_A^2\cos^2\theta)}}{2} = v_1^2, v_2^2 \tag{4.16}$$

(where v_1, v_2 are associated with the plus and minus signs, respectively), Eq. (4.15) can further be written in the form

$$\begin{aligned}&(v_0^2 - v_1^2)(v_0^2 - v_2^2)\\ &= -\frac{\epsilon(iv_0k + 1)}{(1 + v_0^2k^2)}\left[(v_0^2 + c_A^2\cos^2\theta)\,ic_e^2kv_0 + (v_0^2 - v_1^2)(v_0^2 - v_2^2)\right],\end{aligned}$$

which yields (when v_0^2 is replaced by $v_1^2 + \bar{\epsilon}$, or by $v_2^2 + \bar{\epsilon}$, and higher order terms in $\bar{\epsilon}$ (and also $\bar{\epsilon}\epsilon$) are neglected)

$$v_0^2 = v_1^2 + \frac{\epsilon c_e^2 k^2 v_1^2 (v_1^2 + c_A^2 \cos^2\theta)}{(1 + k^2 v_1^2)(v_1^2 - v_2^2)} - i\,\frac{\epsilon k c_e^2 v_1 (v_1^2 + c_A^2 \cos^2\theta)}{(1 + k^2 v_1^2)(v_1^2 - v_2^2)}, \tag{4.17}$$

$$v_0^2 = v_2^2 + \frac{\epsilon c_e^2 k^2 v_2^2 (v_2^2 + c_A^2 \cos^2\theta)}{(1 + k^2 v_2^2)(v_2^2 - v_1^2)} - i\,\frac{\epsilon k c_e^2 v_2 (v_2^2 + c_A^2 \cos^2\theta)}{(1 + k^2 v_1^2)(v_2^2 - v_1^2)}. \tag{4.18}$$

For drawing the velocity phase diagram we require the real parts of v_0. Accordingly we write the real parts of v_0 in the following form (for details see Appendix C)

$$v_0 = c_A \left(\frac{3 + \sqrt{9 - 8\cos^2\theta}}{2}\right)^{1/2} \times \left[1 + \frac{\epsilon\delta(3 + 2\cos^2\theta + \sqrt{9 - 8\cos^2\theta})}{\{\delta(3 + \sqrt{9 - 8\cos^2\theta}) + 2\}\sqrt{9 - 8\cos^2\theta}}\right], \tag{4.19}$$

$$v_0 = c_A \left(\frac{3 - \sqrt{9 - 8\cos^2\theta}}{2}\right)^{1/2} \times \left[1 - \frac{\epsilon\delta(3 + 2\cos^2\theta - \sqrt{9 - 8\cos^2\theta})}{\{\delta(3 - \sqrt{9 - 8\cos^2\theta}) + 2\}\sqrt{9 - 8\cos^2\theta}}\right], \tag{4.20}$$

where $k^2 c_A^2 = \delta$, $c_e = \sqrt{2}\, c_A$, and $\omega_0^2 = c_e^2 + c_A^2$.

5. Discussion of the Results

To obtain the qualitative and quantitative behavior of the phase velocity, we have drawn the wave phase velocity diagram for different values of the parameter $\delta = k^2 c_A^2$ and $\epsilon = 0.35$. The phase velocity plotted on a Friedrichs-type diagram i the real part of v_0 and $0 \leqslant \theta \leqslant \pi/2$ (See Fig. 1 below).

The main points of interest that emerge out of this diagram are:

(i) The 'fast wave' velocity increases with the increase in the values of the non-equilibrium parameter δ and is bounded when δ tends to infinity.

(ii) The increase in the 'fast wave' velocity is maximum when the local magnetic field vector is aligned to the velocity vector.

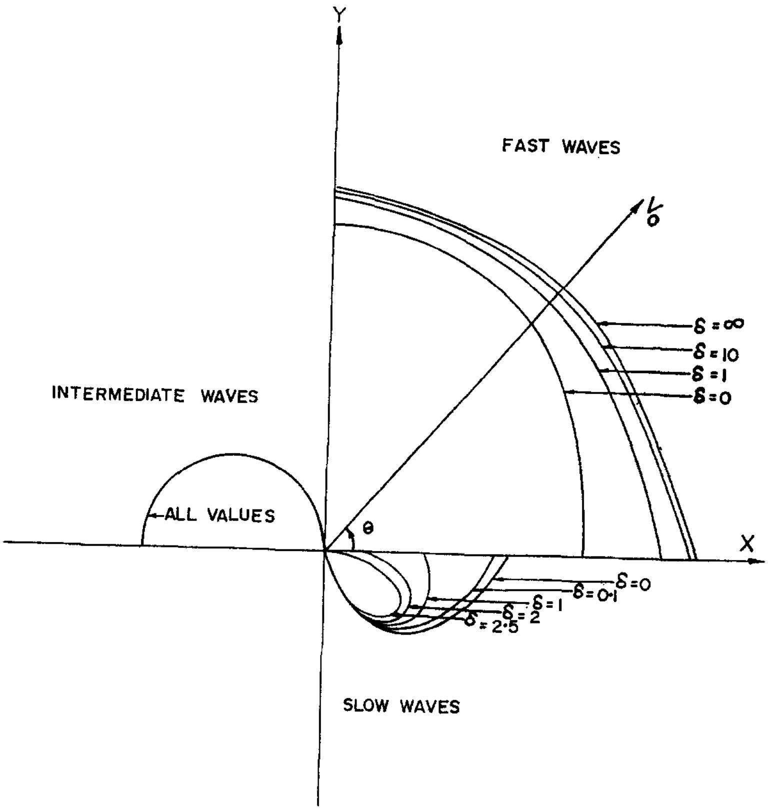
(iii) The intermediate wave remains unaltered.

(iv) The 'slow wave' velocity decreases with the increase in δ.

(v) For $\delta > 2.5$, there is no propagation of the 'slow wave'.

(vi) The rate of diminution of the 'slow wave' speed is greatest when the local magnetic field vector is aligned to the velocity vector.

(vii) For $\theta = \pi/2$, there is only a 'fast wave.'



WAVE-SPEED DIAGRAM (MODIFIED FAST, SLOW AND INTERMEDIATE PHASE SPEEDS VS. WAVE INCLINATION) DRAWN FOR THE CASE $C_e = \sqrt{2}\, C_A$

FIG. 1

If we compare the results obtained here to the case of the Hall current effect on wave propagation (cf. [12]), we find that the two (nonequilibrium effect and Hall effect) are almost the same.

6. General Character of the Various Modes

For a better understanding of the effect of magnetic and nonequilibrium parameters on wave-propagation, one must study the interaction of two sets of waves obtained from (3.18) under the two extreme limits of $k \to 0$ and $k \to \infty$.

To this end, we use Whitham's technique [10]. Using Laplace transform technique, Whitham has studied extensively the following wave equation

$$\left(\frac{\partial}{\partial t} + c_1 \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c_2 \frac{\partial}{\partial x}\right)\phi + \lambda\left(\frac{\partial}{\partial t} + a \frac{\partial}{\partial x}\right)\phi = 0,$$

where c_1, c_2, and a are different wave speeds and λ is a known constant. In particular, he has shown for the above equation and for a small time the higher-order term dominates, while the lower-order term will produce an exponential damping of the wave described by the higher-order term. In turn, the higher order term will produce a diffusion of the wave described by the lower order term.

Application of Whitham's technique to the problem at hand necessitates the reduction of (3.11) to the above form. Consequently, we proceed as follows.

Equation (3.11) can be re-arranged in the form (see (3.14), (4.1a))

$$\begin{aligned}\frac{\partial}{\partial t}\Big[\frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} &- \omega^2 \nabla(\nabla \cdot \boldsymbol{\xi}) \\ &+ c_A^2\{\nabla(\mathbf{b}\cdot\nabla)(\mathbf{b}\cdot\boldsymbol{\xi}) - (\mathbf{b}\cdot\nabla)^2\boldsymbol{\xi} + \mathbf{b}(\mathbf{b}\cdot\nabla)(\nabla\cdot\boldsymbol{\xi})\}\Big] \\ &+ \lambda_0\Big[\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} - \omega_0^2\nabla(\nabla\cdot\boldsymbol{\xi}) \\ &+ c_A^2\{\nabla(\mathbf{b}\cdot\nabla)(\mathbf{b}\cdot\boldsymbol{\xi}) - (\mathbf{b}\cdot\nabla)^2\boldsymbol{\xi} + \mathbf{b}(\mathbf{b}\cdot\nabla)(\nabla\cdot\boldsymbol{\xi})\}\Big] = 0,\end{aligned} \tag{6.1}$$

where

$$\lambda_0 = \frac{c_f^2}{k c_e^2} \geqslant 0. \tag{6.2}$$

Taking the scalar product of (6.1) with ∇ and $\mathbf{b}$ separately and then eliminating $(\mathbf{b}\cdot\boldsymbol{\xi})$ from the resulting equations, we get

$$\begin{aligned}\frac{\partial}{\partial t}&\left[\frac{\partial^4}{\partial t^4} - \omega^2\nabla^2\frac{\partial^2}{\partial t^2} + c_A^2 c_f^2 \nabla^2(\mathbf{b}\cdot\nabla)^2\right]\nabla\cdot\boldsymbol{\xi} \\ &+ \lambda_0\left[\frac{\partial^4}{\partial t^4} - \omega_0^2\nabla^2\frac{\partial^2}{\partial t^2} + c_e^2 c_A^2\nabla^2(\mathbf{b}\cdot\nabla)^2\right]\nabla\cdot\boldsymbol{\xi} = 0.\end{aligned} \tag{6.3}$$

Equation (6.3) can be written as (for waves in the (x, t), plane)

$$\frac{\partial}{\partial t}\left[\frac{\partial^4}{\partial t^4} - \omega^2 \frac{\partial^4}{\partial x^2\,\partial t^2} + c_A{}^2 c_f{}^2 \cos^2\theta \frac{\partial^4}{\partial x^4}\right]\frac{\partial\xi}{\partial x}$$

$$+ \lambda_0\left[\frac{\partial^4}{\partial t^4} - \omega_0{}^2 \frac{\partial^4}{\partial x^2\,\partial t^2} + c_e{}^2 c_A{}^2 \cos^2\theta \frac{\partial^4}{\partial x^4}\right]\frac{\partial\xi}{\partial x} = 0, \qquad (6.3a)$$

where θ is the angle between x-axis and the direction of magnetic field.

Equation (6.3a) can further be written in the form

$$\left[\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} - M\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + M\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - N\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + N\frac{\partial}{\partial x}\right)\right.$$

$$+ \lambda_0\left(\frac{\partial}{\partial t} - M_0\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + M_0\frac{\partial}{\partial x}\right)$$

$$\left.\times\left(\frac{\partial}{\partial t} - N_0\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + N_0\frac{\partial}{\partial x}\right)\right]\frac{\partial\xi}{\partial x} = 0, \qquad (6.4)$$

where

$$M^2 = \tfrac{1}{2}\left[\omega^2 + \sqrt{(\omega^4 - 4c_A{}^2 c_f{}^2 \cos^2\theta)}\right]$$

$$N^2 = \tfrac{1}{2}\left[\omega^2 - \sqrt{(\omega^4 - 4c_A{}^2 c_f{}^2 \cos^2\theta)}\right]$$

$$M_0{}^2 = \tfrac{1}{2}\left[\omega_0{}^2 + \sqrt{(\omega_0{}^4 - 4c_A{}^2 c_e{}^2 \cos^2\theta)}\right]$$

$$N_0{}^2 = \tfrac{1}{2}\left[\omega_0{}^2 - \sqrt{(\omega_0{}^4 - 4c_A{}^2 c_e{}^2 \cos^2\theta)}\right]. \qquad (6.5)$$

From Eq. (6.5), it is quite clear that M, N, M_0 and N_0 can be considered to be positive. Further, for $c_f > c_e$, M, N, M_0 and N_0 satisfy the following inequalities when $0 < \theta < \pi$, $\pi < \theta < 2\pi$

$$-M < -M_0 < -N < -N_0 < 0 < N_0 < N < M_0 < M. \qquad (6.6)$$

The inequalities (6.6) can be verified by a graphical analysis. Further, (6.2) and (6.6) are the necessary conditions for the applicability of Whitham's technique. Since these are satisfied, we give below, without proof, some important results given by that theory.

(i) Under the conditions $\lambda_0 \geqslant 0$ and (6.6), the motion is stable.

(ii) For any nonzero value of λ_0 we have two different sets of waves appearing simultaneously, the first set corresponds to $\lambda_0 \to \infty$ and is given by $-M_0$, M_0, $-N_0$, and N_0 (four real and different speeds), the second corresponds to $\lambda_0 \to 0$ and gives five real and different speeds $-M$, M, $-N$, N and zero (the initial disturbance velocity).

(iii) The principal wave motion is given by lower order terms in Eq. (6.4) but the higher order terms have the role of diffusing these waves.

(iv) The lower order terms in (6.4) have the effect of decaying exponentially the amplitudes of the waves given by the higher order terms.

Our main aim is to determine the effect of the nonequilibrium parameters on wave propagation. Therefore, we confine our interest to the amplitude, decaying factor, and diffusion coefficient of waves.

To examine the assertion (iv) we, following Witham, use the approximation $\partial/\partial t \approx -M(\partial/\partial x)$ in (6.4). This gives, for waves of velocity M, the equation

$$\frac{\partial^4}{\partial x^4}\left[\frac{\partial}{\partial t} + M\frac{\partial}{\partial x} + \frac{\lambda_0(M^2 - N_0^2)(M_0^2 - N_0^2)}{2M^2(M^2 - N^2)}\right]\frac{\partial \xi}{\partial x} = 0, \tag{6.7}$$

whose solution can be written in the form

$$\xi = g\left(t - \frac{x}{M}\right)\exp\left[-\frac{\lambda_0(M^2 - N_0^2)(M_0^2 - N_0^2)}{2M^3(M^2 - N^2)}x\right]. \tag{6.8}$$

Equation (6.8) reveals the damping of waves due to the nonequilibrium state of the medium.

Similarly, for any other wave we can find the amplitude and decaying factor.

For finding the diffusion effect on a lower order wave of velocity M_0, we approximate $\partial/\partial t \approx -M_0(\partial/\partial x)$ in the Eq. (6.4) and this gives

$$\left(\frac{\partial}{\partial t} + M_0\frac{\partial}{\partial x}\right)\bar{\xi} = \frac{(M^2 - M_0^2)(M_0^2 - N^2)}{2\lambda_0(M_0^2 - N_0^2)}\frac{\partial^2 \bar{\xi}}{\partial x^2}, \qquad \bar{\xi} \equiv \frac{\partial^4 \xi}{\partial x^4}. \tag{6.9}$$

Equation (6.9) represents the diffusion of waves of velocity M_0, with diffusion coefficient given by

$$D = \frac{1}{2\lambda_0}\frac{(M^2 - M_0^2)(M_0^2 - N^2)}{(M_0^2 - N_0^2)}. \tag{6.10}$$

From Eq. (6.10), it is clear that nonequilibrium of the gas gives rise to a diffusion effect since otherwise the diffusion coefficient is zero ($\lambda_0 \to \infty$).

For the nonmagnetic case the Eq. (6.10) reduces to

$$D = \frac{kc_e^2}{2c_f^2}(c_f^2 - c_e^2). \tag{6.11}$$

From (6.10), we find that the diffusion coefficient increases with an increase in the strength of the magnetic field.

The other important case that needs particular attention is concerned with the existence of a boundary layer at the initial instance. A detailed analysis of this aspect is dealt with elsewhere.

Appendix A

Here we show how our scheme of working with h (the enthalpy of the gas) instead of s (the specific entropy), which is usually used by Engineers (see Li [14]), is related to Yuan's [11] scheme.

Following Yuan, we can write the energy equation for a gas in which chemical reaction and heat conduction occur in the form (see Eq. (3.1))

$$\rho \frac{Ds}{Dt} = -\frac{1}{T}\nabla \cdot \mathbf{w} + \frac{A}{T}\bar{v}, \tag{A.1}$$

where $\mathbf{w}$ is the heat flux vector, $\bar{v}$ the reaction rate, A the chemical affinity, T the temperature and ρ the density of the gas.

Using (2.7c) of [11] we get

$$\rho \frac{De}{Dt} + \rho p \frac{D\tau}{Dt} = -\nabla \cdot \mathbf{w}. \tag{A.2}$$

Putting $\tau = 1/\rho$ and using the equation of continuity in (A.2) we get

$$\rho \frac{De}{Dt} + p\nabla \cdot \mathbf{v} = -\nabla \cdot \mathbf{w}. \tag{A.3}$$

Equation (A.3) is same as the one derived by Banos [13] for infinitely electrically conducting gas (see Eq. 12 or 17).

Using the relation $h = e + (p/\rho)$ (which is also true in nonequilibrium thermodynamics) and the equation of continuity in (A.3), we get

$$\rho\left(\frac{Dh}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt}\right) = -\nabla \cdot \mathbf{w}. \tag{A.4}$$

For the case of nonheat conducting gas equation (A.4) reduces to (when linearized) Eq. (2.5) of the present paper.

Appendix B

In this section, we outline the approach used by Yuan. In particular, we will show how the use of the phenomenological relations used by Yuan lead us to the rate equations used by Broer (which are also used in the present paper) and to that of Stupochenko and Stakhanov.

Following Prigogine [16], we have the following linear relation between the chemical affinity A and the rate of reaction $\bar{v}$:

$$\bar{v} = \lambda A, \tag{B.1}$$

where λ is a function of the state of the fluid.

Taking the fluid to be isotropic and using Curie's principle and Onsager's symmetric principle, Yuan has shown (see Eq. (2.4)) that the more general relation of the kind given in (B.1) (known as the phenomenological relation for relaxation hydrodynamics) can be reduced to the form

$$\bar{v} = a^{44} \frac{A}{T}, \tag{B.2}$$

where a^{44} is a certain constant.

If $\bar{v}$ be taken equal to Dq/Dt, where q is the progress variable, then the equation (B.1) is similar to the rate equation used by Vincenti; that is,

$$\frac{Dq}{Dt} = \frac{L(p, \rho, q)}{\theta}, \tag{B.3}$$

where L is a function of the thermodynamic state of the medium and θ is a positive quantity related to its specific rate constant.

To obtain Broer's rate equation from (B.2), we expand $A = -\rho(\partial e/\partial q)$ (where e is the internal energy of the reacting gas) in a Taylor series with respect to q, assuming ρ and s constant, about the equilibrium value of reaction, $\bar{q}$, and this gives

$$A = A(\rho, s, \bar{q}) - \rho \left(\frac{\partial^2 e}{\partial q^2}\right)_{\bar{q}} (q - \bar{q}) + \cdots. \tag{B.4}$$

Since $\bar{q}$ is the value of q in equilibrium, $A(\rho, s, \bar{q}) = 0$. Hence Eq. (B.4) gives, to a first approximation,

$$A = -\rho \left(\frac{\partial^2 e}{\partial q^2}\right)_{\bar{q}} (q - \bar{q}). \tag{B.5}$$

It is worth mentioning that in writing (B.4) we have neglected higher powers of $(q - \bar{q})$ and this neglect of higher powers of $(q - \bar{q})$ corresponds to the assumption of a quasi-equilibrium state (as assumed by Broer). Hence (B.2) reduces to

$$\frac{Dq}{Dt} = -\alpha(q - \bar{q}), \tag{B.6}$$

where

$$\alpha = \frac{a^{44}}{T}\left(\frac{\partial^2 e}{\partial q^2}\right)_{\bar{q}}. \tag{B.7}$$

Equation (B.6) is similar to the equation used by Broer, that is

$$\frac{Dq}{Dt} = \frac{q - \bar{q}}{\tau} \tag{B.8}$$

where α (as given in (B.7)) is taken to be a constant and is a valid approximation for the limiting cases of high and low frequencies.

Another approach for arriving at Broer's rate equation from (B.3) (which has been the basis of our study) is available in the work of Vincenti [2].

To obtain the Stupochenko and Stakhanov rate equation from (B.2), we put the value of A or $(-\rho\partial e/\partial q)$ in (B.2) and this gives

$$\bar{v} = -\frac{a^{44}\rho}{T}\left(\frac{\partial e}{\partial q}\right) = -k\frac{\partial e}{\partial q}. \tag{B.9}$$

Equation (B.9) is similar to the rate equation used by Stupochenko and Stakhanov [3] where k is treated as constant (an approximation valid for small disturbance theory).

Appendix C

From Eq. (4.17), we can write

$$v_0 = r^{1/2}\left(\cos\frac{\psi}{2} - i\sin\frac{\psi}{2}\right), \tag{C.1}$$

where

$$r^2 = \left[v_1^2 + \frac{\epsilon c_e^2 k^2 v_1^2(v_1^2 + c_A^2\cos^2\theta)}{(1 + k^2v_1^2)(v_1^2 - v_2^2)}\right]^2 + \frac{\epsilon^2 k^2 c_e^4 v_1^2(v_1^2 + c_A^2\cos^2\theta)^2}{(1 + k^2v_1^2)^2(v_1^2 - v_2^2)^2},$$

and

$$\tan\psi = -\frac{\epsilon\left[\dfrac{kc_e^2(v_1^2 + c_A^2\cos^2\theta)}{(1 + k^2v_1^2)(v_1^2 - v_2^2)}\right]}{v_1\left[1 + \dfrac{\epsilon c_e^2 k^2(v_1^2 + c_A^2\cos^2\theta)}{(1 + k^2v_1^2)(v_1^2 - v_2^2)}\right]}.$$

Neglecting the terms in ϵ^n $(n \geqslant 2)$, we get

$$r^{1/2} = v_1\left[1 + \frac{\epsilon c_e^2 k^2(v_1^2 + c_A^2\cos^2\theta)}{2(1 + k^2v_1^2)(v_1^2 - v_2^2)}\right]. \tag{C.2}$$

Since $\tan\psi \approx 0(\epsilon)$, therefore ψ is small and hence $\cos(\psi/2) \approx 1$.

Therefore, the real part of v_0 reduces to $r^{1/2}$ which is Eq. (4.19). Similarly, from (4.18) Eq. (4.20) can be derived.

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